



RESEARCH ARTICLE

Robustness of analysis of covariance (ANCOVA) under the distributions assumptions and variance homogeneity

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Farklı dağılım ve varyansların homojenliği koşulları altında ANCOVA'nın sağlamlığı

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Öz

Amaç: Tüm parametrik yöntemlerde olduğu gibi, ANCOVA yönteminde de hataların normal dağıldığı, varyansların homojenliği ve hata terimlerinin bağımsız olduğu varsayılmaktadır. Ancak, pratikte, değişkenlere ilişkin dağılımların sıklıkla normal dağılıma uymadığı bilinmektedir. Bu çalışmada, varyansların homojenliği ve farklı dağılım koşulları altında ANCOVA yönteminin tip I hata oranlarının incelenmesi amaçlanmıştır.

Gereç ve Yöntem: Bu amaçla farklı senaryolarda simülasyon çalışmaları yapılmıştır. üç bağımsız grup için birbirine eşit olacak şekilde farklı örneklem büyüklüklerinde Gamma, Beta ve Normal dağılımlardan veri üretimi yapılmıştır. Simülasyon çalışmalarında, gruplar arasındaki farkın anlamlı olmadığı hipotezi altında, 10000 tekrar ile her bir senaryo için tip I hata oranları hesaplanmıştır.

Bulgular: Simülasyon çalışması sonuçlarına göre, homojen varyanslı normal dağılım durumunda, örneklem büyüklüğü $n = 20$ ve $n = 40$ olan gruplarda Tip I hatanın yüksek olduğu bulunmuştur. Heterojen varyans ile normal dağılım durumunda, $n = 10$ ve $n = 30$ ve $n = 40$ örneklem büyüklüğündeki gruplarda sapma gözlemlenmiştir. Bu sonuçlar Gamma dağılımının sonuçları ile aynıdır. Beta dağılımında iki farklı senaryo incelenmiştir. Bunlar dağılım grafiklerinin "U" ve "ters U" biçimlerinde gözlemlendiği durumlardır ve $n = 10$ ve $n = 20$ gibi küçük örneklem büyüklüğünde sapmalar gözlemlenmiştir.

Öneri: Sonuçlar, tip I hata oranının, dağılımın çarpıklığı, örneklem büyüklüğü ve varyansın homojenliği gibi faktörlerden etkilendiğini göstermiştir. Farklı dağılımlar ve parametre değerleri için gerçekleştirilecek simülasyon çalışmaları ile sonuçlar genişletilebilir.

Anahtar kelimeler: Kovaryans analizi, beta, gamma, normal, dayanıklılık, tip I hata

Abstract

Aim: As in all parametric methods, the ANCOVA method assumes that normal distributions of errors, homogeneity of variances, and error terms are independent of each other. However, unusual distributions in practice are more common than normal distribution. In this study, it is aimed to examine ANCOVA method or type 1 error rates under different distribution conditions and homogeneity of variances.

Materials and Methods: For this purpose, a simulation studies under different scenarios was conducted. Random numbers were generated from Gamma, Beta and Normal distributions considering different groups and different sample sizes. In the simulation studies, 10000 replications were run under the null hypothesis of no group differences and type-I error rates were calculated for each scenario.

Results: According to the results, in the case of the normal distribution with homogeneous variance, the proportion of Type I error is high in the groups with the sample size of $n=20$ and $n=40$. In the case of normal distribution with the heterogeneous variance, the deviation has been observed in the groups with the sample size of $n = 10$ and $n = 30$, and $n = 40$. These results are the same as the results of Gamma distribution. In the Beta distribution, there is a deviation in the groups with $n=10$ and $n=20$ where the sample sizes are small.

Conclusion: The results showed that type-I error rate is affected by skewness of the distribution, sample size and homogeneity of variance. Further work can be extended by simulation studies under different distributions and parameter values.

Keywords: Analysis of covariance, beta, gamma, normal, robustness, type-I error





Introduction

The analysis of covariance (ANCOVA) is an important model defined as a combination of the regression analysis and the variance analysis (ANOVA). The ANCOVA is used to test the equality of differences occurring randomly in one or more covariates in two or more groups. The feature of ANCOVA is to increase the power of variance analysis by setting the covariates. In the case of one or more covariates, the ANCOVA model reduces the variability of the random error associated with covariates. This leads to more accurate estimates and robust tests (Actas and Şenoğlu 2018). If the covariates have a strong correlation with the output variable, the ANCOVA will have a lower error variation and may be stronger than the ANOVA for the constant sample size and the same treatment effect sizes (Shieh 2017).

The assumptions of ANCOVA can be listed as follows: (i) The error terms have a normal distribution with an average of zero and a variance of σ^2 . (ii) The variances of error terms are homogeneous. (iii) The error terms are independent of each other. (iv) The relationship between the covariates and the dependent variable is linear. (v) The slopes of the regression lines are homogeneous. In practice, however, these assumptions may not always be ensured. Nevertheless, the violation of one or more of its assumptions may threaten the validity of ANCOVA's results and may require the use of another test (Rheinheimer and Penfield 2001). From these assumptions, the cases where the prerequisites for normality and homogeneity of variances could not be ensured were selected as the focus of this study. These assumptions are the most important assumptions required for the validity of statistical tests, and they are very suitable for evaluation with the simulation studies (Elashoff 1969).

The ANCOVA is widely used in applied sciences to obtain more robust analysis, especially by adjusting the effect of covariates. However, due to the growing number of practical

and ethical concerns associated with the randomization in human sciences, the ANCOVA is now seen as a way to control or adjust the selection bias in an experimental, non-uniform group design (Colliver and Markwell 2006). The aim of this study is to examine the robustness of ANCOVA method in the cases in which the assumptions of normal distribution and homogeneity of variances are violated.

In the literature, there are studies regarding different scenarios for the cases where the assumptions of normal distribution and homogeneity of variances are not ensured. The most well-known of these studies have tested for the scenarios regarding different sample sizes, different prevalence parameters and different kurtosis-skewness parameters, but most of them are based on the normal distribution. In this study, the Gamma and Beta distributions have also been examined in addition to the Normal distribution, and the scenarios for different parameters have been simulated with the sufficient (10,000) replication for all three distributions. In accordance with the literature, the robustness of the results and test statistics has been interpreted in terms of the Type I errors.

The organization of this study is as follows: In section 2, the ANCOVA model has been introduced and the information regarding the scales used and the scenario of the study has been revealed. In section 3 the simulation results and their closeness to the nominal value have been shown by the table and graphical method. In section 4, the similar studies derived from the result of the literature review have been presented and the results have been discussed.

Material and Methods

In this study, various scenarios for Normal, Gamma and Beta distributions have been created in the cases where the variances are constant ($\sigma_1^2 = \sigma_2^2 = \dots = \sigma_g^2$) and increasing ($\sigma_1^2 < \sigma_2^2 < \dots < \sigma_g^2$).

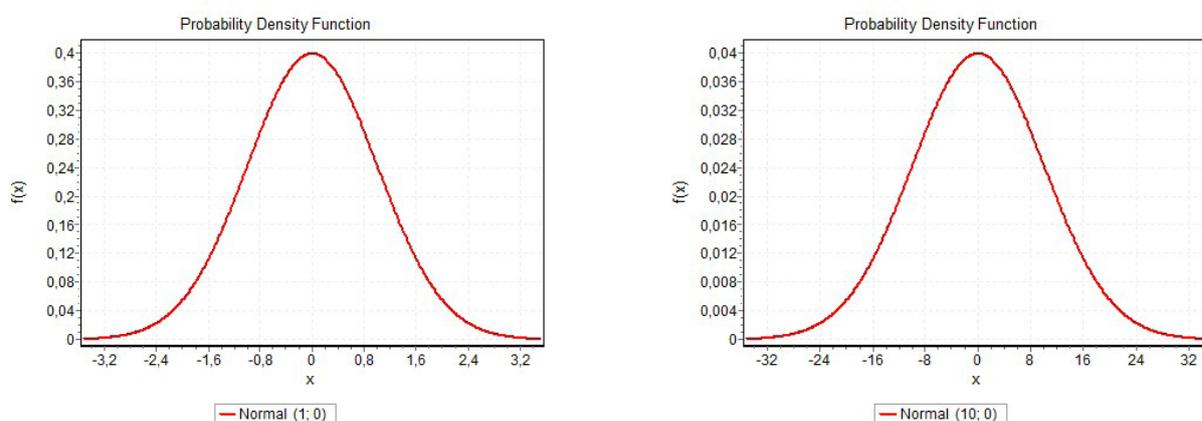


Figure 1. Probability density functions for normal distribution

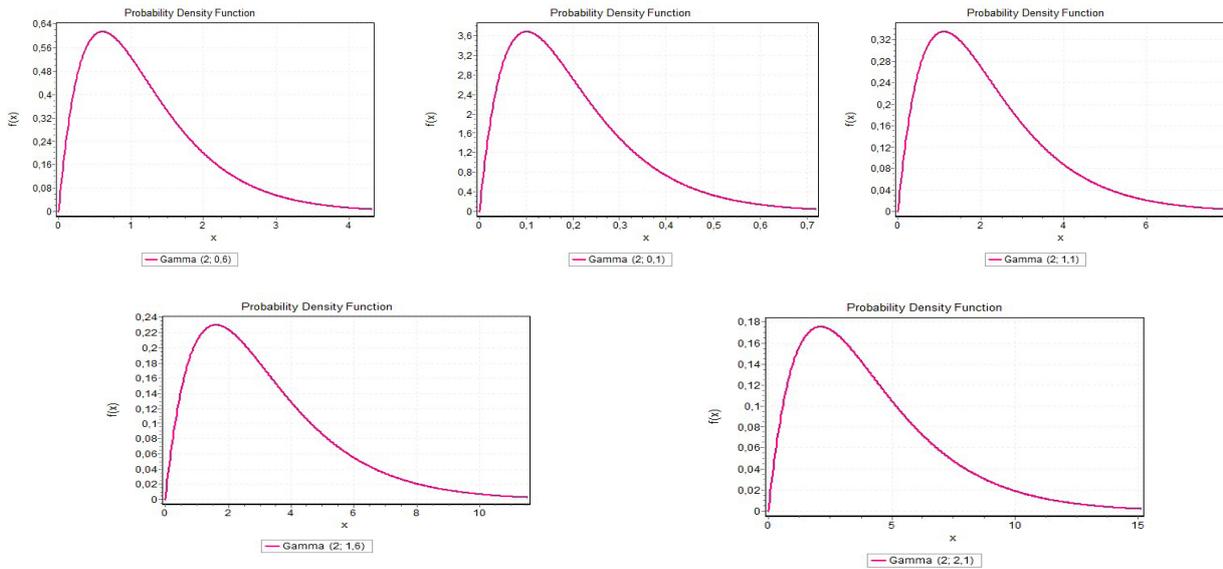


Figure 2. Probability density functions for gamma distribution

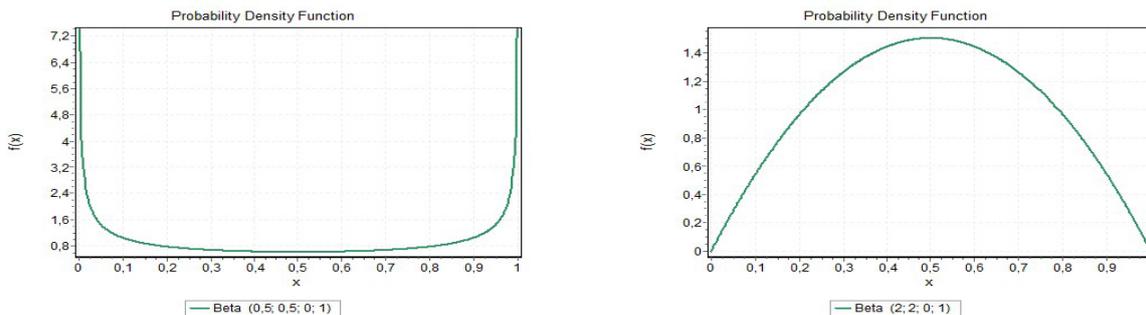


Figure 3. Probability density Functions for beta distribution

The random numbers have been generated for the cases where the number of groups and variables are 3 (three) for these three distributions with different parameter values and different sample sizes ($n=10,20,30,40,50$). The shape and scale parameters for Gamma distribution (2,0.1); (2,0.6); (2,1.1); (2,1.6); (2,2.1)(figure 2); those parameters of (0,1) (0,2)... (0,10) for the normal distribution (figure 1), and for the shape and scale parameters of (0.5,0.5); (2,2) for Beta distribution (figure 3) have been considered. Three groups have been formed in equal sample sizes. One covariate was used for all cases and its distribution remained as normal throughout the study. All distributions and probability density functions related to the distributions are given in figure 1,2 and 3 with their related arguments.

In the simulation studies, 10.000 repetitions have been realized and $\alpha= 0.05$, and the Type-I error values have been calculated for each test. In the simulation study, the calculations have been carried out by using the R Studio program language (version 3.5.0) [CAR [Companion to Applied Regression Functions to Accompany J. Fox and S. Weisberg, An R Companion to Applied Regression, Third Edition, Sage, 2019.] and `rnorm ()`, `rgamma ()` and `rbeta ()` functions.

Model of ANCOVA

The covariance model is obtained by combining the regression and variance analysis models. In equation (1), the regression model is given, and in equation (2), the variance analysis model is presented.

In the case of a possible correlation between Y and Z variables, the regression model is written as follows:

$$Y_{ij} = \mu + \gamma(Z_{ij} - \bar{z}_i) + e_{ij} \quad i=1, \dots, g; \quad j=1, \dots, n_i \quad (1)$$

The variance analysis model is written as;

$$Y_{ij} = \mu + \alpha_i + e'_{ij} \quad i=1, \dots, g \quad ; \quad j=1, \dots, n_i \quad (2)$$

where γ is the actual linear regression coefficient or slope between Z and Y over all data; e_{ij} is the error term; \bar{z} is the average of observation values of Z_{ij} ; g is the number of groups; n_i is the number of units in the (i)th group.

By combining the variance analysis model and the regression



model, the covariance model is written as the equation (3):

$$Y_{ij} = \mu + \gamma(Z_{ij} - \bar{z}_i) + \alpha_i + e_{ij}^* \quad i = 1, \dots, g; j = 1, \dots, n_i \quad (3)$$

where e_{ij}^* , the error term, is smaller than the e_{ij} in the single-factor model due to the elimination of the effect of covariate Z.

In the analysis of covariance model, μ and α_i represent the variance analysis, and γ represents the regression analysis.

The covariance analysis model can be written as belonging to the unit, the group.

The general expression of the regression equation of the $(j)^{th}$ unit in the $(i)^{th}$ unit is as follows (Şahin 2006).

$$y_{ij} = \mu_i + \gamma_i z_{ij} + e_{ij} \quad i = 1, \dots, g; j = 1, \dots, n_i \quad (4)$$

Results

In Table 1, we have observed Type-I error rates of the test statistics obtained from the result of the simulation in which the parameter values of the Normal distribution were (0,1;0,2;0,3;0,4;0,5) and the number of samples in each group was equal and the variances in each group were homogeneous.

In the cases of the normal distribution parameters of N(0,1) with the sample size of n=20-20-20, the normal distribution parameters of N(0,3) with the sample size of n=10-10-10, the normal distribution parameters of N(0,4) with the sample size of n=50-50-50, the normal distribution parameters of N(0,5) with the sample size of n=40-40-40, the test statistics produced the largest deviations from the nominal value. In N (0,1), N (0,5), type 1 error is more liberal. In N (0,1), N (0,5) the type 1 error condition is more conservative. In Figure 1, the deviations from the Type-I error value were expressed visually.

Table 1. Type-I error rates of simulation results for Normal distribution with homogeneous variances and equal sample sizes

n	N(0,1)	N(0,2)	N(0,3)	N(0,4)	N(0,5)
10-10-10	0.0511	0.0494	0.0476	0.0504	0.053
20-20-20	0.0542	0.0516	0.0490	0.0499	0.0512
30-30-30	0.0519	0.0495	0.0499	0.0499	0.0523
40-40-40	0.0516	0.0498	0.0505	0.0503	0.0561
50-50-50	0.0500	0.0513	0.0483	0.0515	0.0517

Table 2. Type-I error rates of simulation results for Normal distribution with heterogeneous variances and equal sample sizes

n	N(0,1) N(0,1) N(0,2)	N(0,1) N(0,2) N(0,4)	N(0,1) N(0,3) N(0,6)	N(0,1) N(0,4) N(0,8)	N(0,1) N(0,5) N(0,10)
10-10-10	0.0552	0.0495	0.0490	0.0514	0.0519
20-20-20	0.0515	0.0530	0.0486	0.0494	0.0517
30-30-30	0.0545	0.0478	0.0491	0.0499	0.0524
40-40-40	0.0500	0.0505	0.0521	0.0510	0.0551
50-50-50	0.0502	0.0518	0.0495	0.0492	0.0523

Table 2 shows the Type-I error rates of the test statistics obtained from the result of the simulation for equal sample sizes, Normal distribution with heterogeneous variances for each group. The variances of third groups were twice the variances of the second group. In other words, the variance for each group was calculated to be twice the previous one.

According to the results, it has been observed that, in the groups with the sample size of $n=10$ and $n=30$ for the case of the distribution parameters of $N(0,1), N(0,1), N(0,2)$, and in the group with the sample size of $n=40$ for the case of $N(0,1), N(0,5), N(0,10)$, the deviation from the nominal value was higher than those in other groups. As a result, for $N(0,1) N(0,1) N(0,2)$ and $N(0,1) N(0,2) N(0,4)$ scenarios, it can be called more liberal. $N(0,1) N(0,3) N(0,6)$ and $N(0,1) N(0,4) N(0,8)$ scenarios were conservative and $N(0,1) N(0,5) N(0,10)$ were found to liberal.

In Figure 2, the deviations from the Type-I error value were expressed visually.

Table 3 indicates the Type-I error rates of the test statistics

obtained from the result of the simulation for Gamma distribution in the case in which the variances in each group were heterogeneous (the variances are monotonous ascending ($\sigma_1 < \sigma_2 < \dots < \sigma_k$)). According to the results, it has been seen that, in the groups with the sample size of $n=10$ and $n=30$ for the case of the distribution parameters of $\text{Gamma}(2,0.1)$, and in the group with the sample size of $n=40$ for the case of $\text{Gamma}(2,2.1)$, the deviation from the nominal value was found to be higher, compared to those in other cases. In terms of gamma distribution, it was observed that the scenarios $(2,0.6)$, $(2,1.1)$ and $(2,1.6)$ were conservative and the others were more liberal. So, in Figure 3, the deviations from the Type-I error value were expressed visually.

Table 4 shows the Type-I error rates of the test statistics obtained from the result of the simulation for Beta distribution in the case where the variances in each group were heterogeneous. It has been observed that, in the groups with the sample size of $n=10$ and $n=20$ for the case of the distribution parameters of $\text{Beta}(0.5,0.5)$, and in the group with the sample size of $n=10$ for the case of $\text{Beta}(2,2)$, the deviation from the nominal value was higher than those in other cases. So, for the gamma distribution, which the number of subjects

Table 3. Type-I error rates of simulation results for Gamma distribution with heterogeneous variances. The number of subjects in the groups is equal, the variances are monotonous ascending ($\sigma_1 < \sigma_2 < \dots < \sigma_k$)

n	Gamma (2, 0.1)	Gamma (2, 0.6)	Gamma (2, 1.1)	Gamma(2, 1.6)	Gamma(2, 2.1)
10-10-10	0.0552	0.0495	0.0490	0.0514	0.0519
20-20-20	0.0515	0.0530	0.0486	0.0494	0.0517
30-30-30	0.0545	0.0478	0.0491	0.0499	0.0524
40-40-40	0.0500	0.0505	0.0521	0.0510	0.0551
50-50-50	0.0502	0.0518	0.0495	0.0492	0.0523

Table 4. Type-I error rates of simulation results for Beta distribution with heterogeneous variances

n	Beta (0.5, 0.5)	Beta (2, 2)
10-10-10	0.0551	0.0557
20-20-20	0.0533	0.0507
30-30-30	0.0520	0.0475
40-40-40	0.0500	0.0501
50-50-50	0.0481	0.0498



in the groups was equal and the variances were planned in monotone increasing ($\sigma_1 < \sigma_2 < \dots < \sigma_k$), in both «U» and «reverse U» cases, deviations from type 1 error are greater in small samples. When the results of Table 4 are summarized, it is observed that the results were liberal in Beta (0.5,0.5) scenario and conservative in Beta (2,2). In Figure 4, the deviations from the Type-I error value were expressed visually.

Discussion

In this study, the simulation exercise have been carried out for different sample sizes and different distributions in the cases in which the ANCOVA assumptions, especially normality and homogeneous variance assumptions, are violated. The Type I error derived has been discussed by comparing with the nominal value and the results of similar studies in the literature. Box and Anderson (1962) analytically examined the effect of violating the conditional normality assump-

tion, and concluded that when the covariate was normally distributed, those without the conditional normality had little effect on Type I errors. However, they stated that when the covariate had a non-normal distribution, the F test was susceptible to the deviations from the conditional normality.

Levy (1980) examined the effect of non-normal conditional distributions (i.e., uniform, double exponential, transformed exponential and transformed chi-square) on the rates of Type I errors. In his simulation, the covariate has the same distribution as the errors. Its results indicated that for both equal and unequal sample sizes, a non-normal variable and non-normal conditional distributions did not significantly affect the rates of Type I errors. Levy's study did not take into account the effect of conditional normality on the power of statistical analysis. When examining the homogeneity of the variances in his study, Potthoff (1965) stated that the robustness of ANCOVA was dependent on the sample size and va-

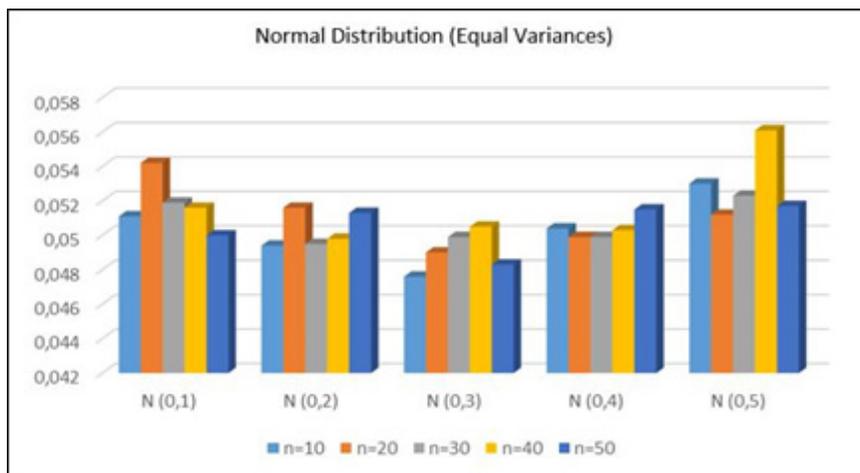


Figure 4. Type-I error rates of simulation results for Normal distribution

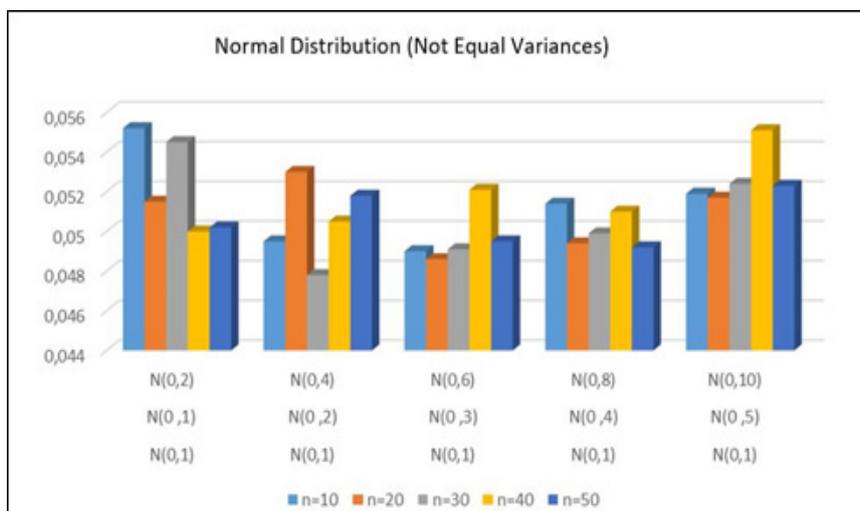


Figure 5. Type-I error rates of simulation results for Normal distribution with heterogeneous variance



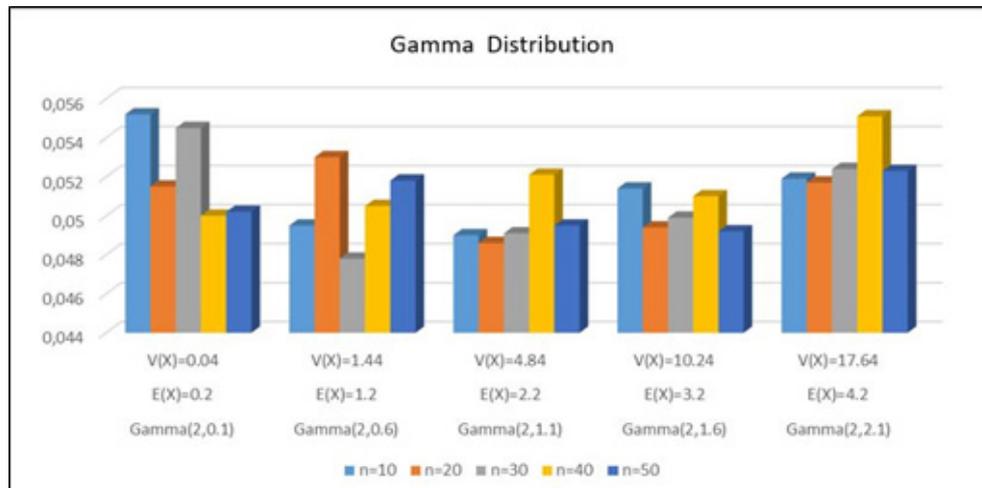


Figure 6. Type-I error rates of simulation results for Gamma distribution with heterogeneous variance

riance of covariates in the groups. In other words, when the sample sizes are equal and the variance of the covariates is the same in groups, the ANCOVA is robust against the violation of this assumption. Shields (1978) showed that in the case when the assumptions were violated, the parametric ANCOVA was robust if the sample sizes were equal. Shields said that when the sample sizes were not equal, the parametric ANCOVA was affected by the violation of assumptions (not robust). However, Shields did not investigate the issue of statistical power in his research. In the study of Olejnik and Algina (1984), they stated that in the case of a violation of both normality and homogeneous variance assumptions, the parametric ANCOVA was not robust for the small samples, and it produced the results below the nominal value. Johnson and Rakow (1994) pointed out that the combination of unequal group variances, sample sizes and regression slopes constituted the biggest threat to the ANCOVA's robustness. In the study of Rheinheim and Penfield (2001), the ANCOVA F test was found to be robust for the balanced designs, but the non-parametric alternative methods generated better results for the unbalanced designs in the case in which the variance was not homogeneous and the sample size was large. D'Alonzo (2004) pointed out that, in the case of the large samples and equal group numbers, the ANCOVA was found to be remained robust, and in the case where the assumption regarding the homogeneity of the regression slopes was violated, the Johnson-Neyman technique might be the most powerful alternative to the ANCOVA. Wilcox (2017) stated that the violation of the two assumptions led to a problem in practice. According to Wilcox, the control of this test over the possibility of Type I error diminishes in the case of a violation of the assumptions

Conclusion

According to the simulation results of this study, in the case of the normal distribution with homogeneous variance, the proportion of Type I error was high in the groups with the

sample size of $n=20$ and $n=40$. In the case of normal distribution with the heterogeneous variance, the deviation was observed in the groups with the sample size of $n = 10$ and $n = 30$, and $n = 40$. These results were the same as the results of Gamma distribution. In other words, the Gamma distribution also showed the same deviations in the same sample sizes. According to these results, regardless of the distribution, it can be claimed that ANCOVA was not robust when the assumption of the homogeneity of variances was not ensured. In the Beta distribution, two cases were examined. These were cases where the distribution graphs had the shapes of 'U' and 'reverse U'. In this distribution, there was a deviation in the groups with $n=10$ and $n=20$ where the sample sizes are small. In the case when the sample size was small and the assumptions of the normality and homogeneous variance were violated, the ANCOVA was not robust. In general, the results had indicated that the proportion of Type I error was affected by the skewness of the distribution, the sample size and the homogeneity of the variances. In general, it was concluded that if the variation is high and the sample size is small, the results found to be liberal, and for the cases which the variation is low and the sample size is large, the results were found to be conservative. In the cases in which the ANCOVA was not robust, the common opinion of the use of non-parametric methods was frequently witnessed within the literature.

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Conflict of Interest

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References

- Acitas, S , Senoglu, B . Robust factorial ANCOVA with LTS error distributions. Hacettepe Journal of Mathematics and Statistics, (2018);47 (2), 347-363. Patterson, R.F. &
- Box, GEP, Muller, ME. A note on the generation of random normal deviates. *Annals of Mathematical Statistics*, (1958);28, 610-611.
- Colliver, JA., Markwell SJ. ANCOVA, Selection Bias, Statistical Equating, and Effect Size:Recommendations for Publication. *Teaching and Learning in Medicine*, (2006);18(4), 284–286.
- D'Alonzo, KT. The Johnson-Neyman Procedure as an Alternative to ANCOVA. *West J Nurs Res*. (2004);26(7): 804–812.
- Elashoff, JA. Analysis of covariance: A delicate instrument. *American Educational Research Journal*, (1969);6(3), 383-401.
- Johnson, CC, Rakow, EA. (1994). Effects of violations of data set assumptions when using the analysis of variance and covariance with unequal group sizes. Paper presented at the annual meeting of the Mid-South Educational Research.
- Levy, K. J. A monte carlo study of analysis of covariance under violations the assumptions of normality and equal regression slopes. *Educational and Psychological Measurement*, (1980); 40, 835-840.
- Olejnik, S. F., & Algina, J. Parametric ANCOVA and the rank transform ANCOVA when the data are conditionally non-normal and heteroscedastic. *Journal of Educational Statistics*, (1984);9(2), 129-149.
- Potthoff, A. F. Some Scheffe-type tests for some Behrens-Fisher type regression problems. *Journal of the American Statistical Association*, (1965); 60, 1163-1190.
- Rheinheimer, DC., Penfield, AD. The Effects of Type I Error Rate and Power of the ANCOVA F Test and Selected Alternatives under Nonnormality and Variance Heterogeneity. *The Journal of Experimental Education*, (2001); 69(4), 373-391.
- Shieh, G. Power and Sample Size Calculations for Contrast Analysis in ANCOVA. *Multivariate Behavioral Research*, (2017); 52(1), 1-11.
- Shields, JL. (1978). An empirical investigation of the effect of heteroscedasticity and heterogeneity of variance on the analysis of covariance and the Johnson-Neyman technique (Tech. Rep. No. 292). Alexandria, VA: U.S. Army Research Institute for the Behavioral and Social Sciences.
- Wilcox, R. Robust ANCOVA: Confidence intervals that have some specified simultaneous probability coverage when there is curvature and two covariates. *Journal of Modern Applied Statistical Methods*, (2017); 16(1), 3-19.

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